

The descriptive set-theoretic complexity of the set of points of continuity of a multi-valued function (Extended Abstract)

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In this article we treat a notion of continuity for a multi-valued function F and we compute the descriptive set-theoretic complexity of the set of all x for which F is continuous at x . We give conditions under which the latter set is either a G_δ set or the countable union of G_δ sets. Also we provide a counterexample which shows that the latter result is optimum under the same conditions. Moreover we prove that those conditions are necessary in order to obtain that the set of points of continuity of F is Borel i.e., we show that if we drop some of the previous conditions then there is a multi-valued function F whose graph is a Borel set and the set of points of continuity of F is not a Borel set. Finally we give some analogue results regarding a stronger notion of continuity for a multi-valued function. This article is motivated by a question of M. Ziegler in [*Real Computation with Least Discrete Advice: A Complexity Theory of Nonuniform Computability with Applications to Linear Algebra*, submitted].

1 Introduction.

A *multi-valued* function F from a set X to another set Y is any function from X to the power set of Y i.e., F assigns sets to points. Such a function will be denoted by $F : X \Rightarrow Y$. A multi-valued function $F : X \Rightarrow Y$ can be identified with its *graph* $Gr(F) \subseteq X \times Y$ which is defined by

$$(x, y) \in Gr(F) \iff y \in F(x).$$

This way we view F as a subset of $X \times Y$. From now on we assume that all given multi-valued functions are between metric spaces and that they are *total* i.e., if $F : X \Rightarrow Y$ is given then $F(x) \neq \emptyset$ for all $x \in X$, in other words the projection of F along Y is the whole space X .

There are various notions of continuity for multi-valued functions, here we focus on two of those (see [2] Definition 2.1, [3] pp. 70-71 and [1] p. 82, p. 93).

Definition 1.1. Let (X, p) and (Y, d) be metric spaces; a multi-valued function $F : X \Rightarrow Y$ is *continuous at x* if there is some $y \in F(x)$ such that for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all $x' \in B_p(x, \delta)$ there is some $y' \in F(x')$ for which we have that $d(y, y') < \varepsilon$.

*The author is currently a post-doctoral researcher at the Mathematics Department of TU Darmstadt in the work-group of ULRICH KOHLENBACH (TU Darmstadt), whom the author would like to thank. The author owns special thanks to MARTIN ZIEGLER (TU Darmstadt) for bringing to the author's attention the problem which motivated this article and for his valuable advice. Finally the author would like to thank the referees of this article for their kind comments and suggestions.

Definition 1.2. Let (X, p) and (Y, d) be metric spaces; a multi-valued function $F : X \rightrightarrows Y$ is *strongly continuous at x* if for all $y \in F(x)$ and for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all $x' \in B_p(x, \delta)$ there is some $y' \in F(x')$ for which we have that $d(y, y') < \varepsilon$.

It is clear that both these notions generalize the classical notion of continuity of functions. Moreover it is also clear that the continuity/strong continuity of a multi-valued function is preserved under distance functions which generate the same topology.

The motivation of this article is the following question posed by M. Ziegler in [6] (Question 59(a)). It is well known that if we have a function $f : (X, p) \rightarrow (Y, d)$ then the set of points of continuity of f is a G_δ subset of X ; see for example 3.B in [4]. So what can be said about the descriptive set-theoretic complexity of the set of points of continuity/strong continuity of a multi-valued function $F : (X, p) \rightrightarrows (Y, d)$? In this article we present the answer for the case of continuity and some analogue results for the case of strong continuity. The full answer for the latter case is still under investigation.

We proceed with the basic terminology and notations. By ω we denote the set of natural numbers (including the number 0). Suppose that X and Y are two topological spaces. We call a function $f : X \rightarrow Y$ a *topological isomorphism* between X and Y if the function f is bijective, continuous and the function f^{-1} is continuous. We will also say that the space X is *topologically isomorphic* with Y if there exists a topological isomorphism between X and Y .

The *Baire space* \mathcal{N} is the set of all sequences of naturals i.e., $\mathcal{N} = \omega^\omega$ with the usual product topology. We call the members of the Baire space as *fractions* and we usually denote them by lower case Greek letters α, β etc. One choice of basic neighborhoods for the product topology on \mathcal{N} is the collection of the following sets

$$N(k_0, \dots, k_{n-1}) = \{\alpha \in \mathcal{N} \mid \alpha(0) = k_0, \dots, \alpha(n-1) = k_{n-1}\}$$

where $k_0, \dots, k_{n-1} \in \omega$. The set of ultimately constant sequences is clearly countable and dense in \mathcal{N} ; thus the latter is a separable space. For $\alpha, \beta \in \mathcal{N}$ with $\alpha \neq \beta$ define

$$d_{\mathcal{N}}(\alpha, \beta) = 1/(\text{least } n [\alpha(n) \neq \beta(n)] + 1).$$

Also put $d_{\mathcal{N}}(\alpha, \alpha) = 0$ for all $\alpha \in \mathcal{N}$. It is not hard to see that the function $d_{\mathcal{N}}$ is a complete distance function on \mathcal{N} which generates its topology. From now on we think of the Baire space \mathcal{N} with this distance function $d_{\mathcal{N}}$.

We denote by \mathcal{C} the subset of the Baire space \mathcal{N} which consist of all sequences which values 0 and 1 i.e., $\mathcal{C} = 2^\omega$. The set \mathcal{C} with the induced topology is a compact space. It is not hard to see that \mathcal{C} is topologically isomorphic with the usual Cantor set of the unit interval. This result motivates us to call \mathcal{C} as the *Cantor space*.

We denote by ω^* the set of all finite sequences of ω . If $u \in \omega^*$ then there are unique naturals n, k_0, \dots, k_{n-1} such that $u = (k_0, \dots, k_{n-1})$. The *length* of u is the previous natural n and we denote it by $lh(u)$. Also we write $u(i) = k_i$ for all $i < lh(u)$, so that $u = (u(0), \dots, u(lh(u) - 1))$. It is convenient to include the *empty sequence* in ω^* i.e., the one with zero length. The latter will be denoted by $\langle \cdot \rangle$. So when we write $u = (u(0), \dots, u(n-1))$ we will always mean in case where $n = 0$ that $u = \langle \cdot \rangle$. If $u \in \omega^*$ and $n \in \omega$ we denote the finite sequence $(u(0), \dots, u(lh(u) - 1), n)$ by $u^\wedge(n)$. We write $u \sqsubseteq v$ exactly when $lh(u) \leq lh(v)$ and $u(i) = v(i)$ for all $i < lh(u)$ i.e., $u \sqsubseteq v$ means that v is an extension of u or equivalently u is an initial segment of v .

A set $T \subseteq \omega^*$ is called a *tree* on ω if it is closed under initial segments i.e.,

$$v \in T \ \& \ u \sqsubseteq v \implies u \in T.$$

The members of a tree T are called *nodes* or *branches* of T . A tree T is of *finite branching* if and only if for all $u \in T$ there are only finitely many $n \in \omega$ such that $u^\frown(n) \in T$. A fraction α is an *infinite branch* of T if and only if for all $n \in \omega$ we have that $(\alpha(0), \dots, \alpha(n-1)) \in T$. The *body* $[T]$ of a tree T is the set of infinite branches of T .

For practical reasons when we refer to a tree T we will always assume that T is not empty i.e., $\langle \cdot \rangle \in T$. Define

$$Tr = \{T \subseteq \omega^* \mid \text{the set } T \text{ is a tree on } \omega\}.$$

We may view every $T \in Tr$ as a member of 2^{ω^*} by identifying T with its characteristic function $\chi_T : \omega^* \rightarrow \{0, 1\}$. Since the set ω^* is countable the space 2^{ω^*} with the product topology is completely metrizable - in fact it is topologically isomorphic with the Cantor space \mathcal{C} . Moreover the set Tr is a closed subset of \mathcal{C} . Indeed let $T_i \in Tr$ for all $i \in \omega$ be such that $T_i \xrightarrow{i \rightarrow \infty} S$ for some $S \in 2^{\omega^*}$; we will prove that $S \in Tr$. From the hypothesis it follows that for all $u \in \omega^*$ there is some $i_0 \in \omega$ such that for all $i \geq i_0$ we have that

$$u \in T_i \iff u \in S.$$

Taking $u = \langle \cdot \rangle$ since $T_i \in Tr$ for all $i \in \omega$ we have that $\langle \cdot \rangle \in S$ and so S is not empty. Also if $u, v \in \omega^*$ we find i large enough so that $u \in T_i \iff u \in S$ and $v \in T_i \iff v \in S$. So if $u \in S$ and $v \sqsubseteq u$ then $u \in T_i$ and since T_i is a tree we also have that $v \in T_i$; hence $v \in S$. Therefore $S \in Tr$ and the set of trees Tr is closed in \mathcal{C} .

We make a final comment about trees. For any non-empty set S of finite sequences of naturals the tree T which is *generated* by S is the following

$$\{u \mid (\exists w \in S)[u \sqsubseteq w]\}$$

i.e., the tree which is generated by S is the tree which arises by taking all initial segments of members of S .

Suppose that X is a metric space. The family $\Sigma_1^0(X)$ is the collection of all *open* subsets of X . Inductively we define the family $\Sigma_{n+1}^0(X)$ for $n \geq 1$ as follows: for $A \subseteq X$,

$$A \in \Sigma_{n+1}^0(X) \iff A = \bigcup_{i \in \omega} A_i, \text{ where } X \setminus A_i \in \Sigma_{k_i}^0(X) \text{ for some } k_i \leq n \text{ for all } i \in \omega.$$

Put also

$$\Pi_n^0(X) = \{B \subseteq X \mid X \setminus B \in \Sigma_n^0(X)\}$$

and $\Delta_n^0(X) = \Sigma_n^0(X) \cap \Pi_n^0(X)$ for all $n \geq 1$. Notice that family $\Pi_1^0(X)$ is the collection of all closed subsets of X , the family $\Sigma_2^0(X)$ is the collection of all F_σ subset of X and so on. By a simple induction one can prove that $\Sigma_n^0(X) \cup \Pi_n^0(X) \subseteq \Delta_{n+1}^0(X)$ for all $n \geq 1$. It is well known that in case where X admits a complete distance function and it is an uncountable set then $\Sigma_n^0(X) \neq \Pi_n^0(X)$ for all $n \geq 1$ and so the previous inclusion is a proper one for all $n \geq 1$, (see [4] and [5]).

The families $\Sigma_n^0(X), \Pi_n^0(X)$ are closed under finite unions, finite intersections, and continuous pre-images i.e., if $f : X \rightarrow Y$ is continuous and $B \subseteq Y$ is in $\Sigma_n^0(Y)$ then $f^{-1}[B]$ is in $\Sigma_n^0(X)$. Moreover it is clear that if $f : X \rightarrow Y$ is a topological isomorphism then for all $A \subseteq X$ we have that $A \in \Sigma_n^0(X)$ if and only if $f[A] \in \Sigma_n^0(Y)$ and similarly for $\Pi_n^0(X)$ for all $n \in \omega$. Finally the family $\Sigma_n^0(X)$ is closed under countable unions, the family $\Pi_n^0(X)$ is closed under countable intersections and the family $\Delta_n^0(X)$ is closed under complements. We usually say that A is in Σ_n^0 when X is easily understood from the context. It is clear that all sets in Σ_n^0 are Borel sets.

We now deal with a bigger family of sets. Suppose that X is separable and that X admits a complete distance function. A set $A \subseteq X$ is in $\Sigma_1^1(X)$ or it is *analytic* if A is the continuous image of a closed subset of a complete and separable metric space.¹ It is well known that in the definition of analytic sets we may replace the term “continuous image” by “Borel image” (i.e., image under a Borel measurable function) and/or the term “closed subset” by “Borel set”, (see [4] and [5]). A set $B \subseteq X$ is in $\Pi_1^1(X)$ or it is *co-analytic* if the set $X \setminus B$ is analytic and B is in $\Delta_1^1(X)$ or it is *bi-analytic* if B is both analytic and co-analytic. It is well known that every Borel set is analytic, hence every Borel set is bi-analytic. A classical theorem of Suslin states that a set B is Borel if and only if it is bi-analytic, (see [5] 2E.1 and 2E.2).

The families $\Sigma_1^1(X)$, $\Pi_1^1(X)$ and $\Delta_1^1(X)$ are closed under countable unions, countable intersections and continuous pre-images. Moreover the family $\Sigma_1^1(X)$ is closed under Borel images i.e., if Y is a complete and separable metric space, $f : X \rightarrow Y$ is a Borel measurable function and A is an analytic subset of X then $f[A]$ is an analytic subset of Y . Finally if X is uncountable we have that $\Sigma_1^1(X) \neq \Pi_1^1(X)$ and in particular there is an analytic set which is not Borel.

We can pursue this hierarchy further by defining the family $\Sigma_{n+1}^1(X)$ as the collection of all subsets of X which are the continuous image of a Π_n^1 subset of a complete and separable metric space. Similarly one defines the family $\Pi_{n+1}^1(X)$ as the collection of all subsets of X whose complement is in $\Sigma_{n+1}^1(X)$ and the family $\Delta_{n+1}^1(X)$ as the collection of all subsets of X which belong both to $\Sigma_{n+1}^1(X)$ and $\Pi_{n+1}^1(X)$. By a simple induction one can prove that $\Sigma_n^1(X) \cup \Pi_n^1(X) \subseteq \Delta_{n+1}^1(X)$ for all $n \geq 1$. Also the analogous properties stated above are true. The reader may refer to [5] for more information on those classes.

The proofs of the forthcoming theorems make a substantial use of techniques of Descriptive Set Theory which involve the use of many quantifiers. Of course those quantifiers can be interpreted as unions and intersections of sets and this is what we usually do in order to prove that a given set is for example Π_2^0 . There are some cases though (for example in the proof of Theorem 2.6) where this interpretation becomes too complicated. In these cases it is better to think of a given set P as a relation in order to derive its complexity. The reader can consult section 1C of [5] on how one can make computations with relations.

2 Results about the set of points of continuity of a multi-valued function.

We begin with some positive results regarding the set of points of continuity of a multi-valued function F . Recall that a topological space Y is *exhaustible by compact sets* if there is a sequence $(K_n)_{n \in \omega}$ of compact subsets of Y such that every K_n is contained in the interior of K_{n+1} and $Y = \bigcup_{n \in \omega} K_n$. Notice the lack of any hypothesis about the set F in the next theorem.

Theorem 2.1. *Let (X, p) and (Y, d) be metric spaces with (Y, d) being separable and let $F : X \Rightarrow Y$ be a multi-valued function.*

- (a) *If the set $F(x)$ is compact for all $x \in X$ then the set of points of continuity of F is Π_2^0 i.e., G_δ .*
- (b) *If Y is exhaustible by compact sets and the set $F(x)$ is closed for all $x \in X$, then the set of points of continuity of F is Σ_3^0 .*

Theorem 2.1 has an interesting corollary which answers Question 59(a) posed by M. Ziegler in [6].

¹The notion of an analytic set can be treated in a more general context of spaces; however we prefer to stay in the context of complete and separable metric spaces.

Corollary 2.2. *Suppose that X is a metric space and that $F : X \Rightarrow \mathbb{R}^m$ is a multi-valued function such that the set $F(x)$ is closed for all $x \in X$.*

- (a) *The set of the points of continuity of F is Σ_3^0 .*
- (b) *If moreover the set $F(x)$ is bounded for all $x \in X$ then the set of points of continuity of F is Π_2^0 .*

We now show that the results of Theorem 2.1 are optimum. It is well known that there are functions $f : [0, 1] \rightarrow \mathbb{R}$ for which the set of points of continuity is not F_σ . Therefore the Π_2^0 -answer is the best one can get. Thus we only need to deal with the Σ_3^0 -answer. The following lemmas, although being straightforward from the definitions, will prove an elegant tool for the constructions that will follow.

Lemma 2.3. *Suppose that (X_0, p_0) , (X_1, p_1) , (Y, d) are metric spaces and that there exists a function $f : X_0 \rightarrow X_1$ such that $f[X_0]$ is closed and $f : X_0 \rightarrow f[X_0]$ is a topological isomorphism. Assume that we are given a multi-valued function $F : X_0 \Rightarrow Y$. Define the multi-valued function $\tilde{F} : X_1 \Rightarrow Y$ as follows:*

$$\tilde{F}(x_1) = F(x_0) \text{ if } x_1 = f(x_0) \text{ for some } x_0 \in X_0 \text{ and } \tilde{F}(x_1) = Y \text{ otherwise.}$$

Then

- 1. *\tilde{F} is continuous at x_1 if and only if either $x_1 \notin f[X_0]$ or $x_1 = f(x_0)$ and F is continuous at x_0 . Hence if we denote by P_0 and P_1 the set of points of continuity of F and \tilde{F} respectively we have that*

$$P_1 = f[P_0] \cup (X_1 \setminus f[X_0]).$$

- 2. *If Γ is any of the classes Σ_n^0, Π_n^0 , with $n \geq 2$ or Δ_1^1 then*

$$P_1 \in \Gamma \iff P_0 \in \Gamma.$$

In particular if the set of points of continuity of F is not Π_3^0 (Borel) then the set of points of continuity of \tilde{F} is not Π_3^0 (Borel respectively).

Moreover the sets F and \tilde{F} as subsets of $X_0 \times Y$ and $X_1 \times Y$ respectively satisfy

$$F \in \Gamma \iff \tilde{F} \in \Gamma.$$

- 3. *If $F(x_0)$ is a closed subset of Y for all $x_0 \in X_0$ then $\tilde{F}(x_1)$ is also a closed subset of Y for all $x_1 \in X_1$.*

Lemma 2.3 has a cute corollary which might be regarded as the multi-valued analogue of the Tietze Extension Theorem.

Corollary 2.4. *Every continuous multi-valued function which is defined on a closed subset of a metric space can be extended continuously on the whole space.*

Lemma 2.5. *Let X, Y, Z be metric spaces, $F : X \Rightarrow Y$ be a multi-valued function and $\pi : Y \rightarrow Z$ be a topological isomorphism between Y and $\pi[Y]$. Define the composition $\pi \circ F : X \Rightarrow Z$:*

$$(\pi \circ F)(x) = \pi[F(x)], \quad x \in X.$$

The following hold.

- 1. *A point $x \in X$ is a point of continuity of F if and only if x is a point of continuity of $\pi \circ F$;*
- 2. *If the set $\pi[Y]$ is closed and the set $F(x)$ is closed for some $x \in X$ then the set $(\pi \circ F)(x)$ is also closed.*
- 3. *If F is a Borel subset of $X \times Y$ then $\pi \circ F$ is a Borel subset of $X \times Z$.*

Theorem 2.6. *There is a multi-valued function $F : [0, 1] \rightarrow \mathbb{R}$ such that the set $F(x)$ is closed for all x , the set F is a Π_2^0 subset of $[0, 1] \times \mathbb{R}$ and the set of points of continuity of F is not Π_3^0 . Therefore the Σ_3^0 -answer is the best possible for a multi-valued function F from $[0, 1]$ to \mathbb{R} even if F is below the Σ_3^0 -level.*

One can ask what is the best that we can say about the set of points of continuity of F without any additional topological assumptions for Y or for $F(x)$. The following proposition gives an upper bound for the complexity of this set. (Notice though that we restrict ourselves to complete and separable metric spaces.)

Proposition 2.7. *Let (X, p) and (Y, d) be complete and separable metric spaces and let $F : X \Rightarrow Y$ be a multi-valued function such that the set $F \subseteq X \times Y$ is analytic. Then the set of points of continuity of F is analytic as well.*

Now we show that if we remove just one of our assumptions about $F(x)$ or about Y in Theorem 2.1, then it is possible that the set of points of continuity of F is not even a Borel set. Therefore Proposition 2.7 is the best that one can say in the general case.

Theorem 2.8.

- (a) *There is a multi-valued function $F : \mathcal{C} \Rightarrow \mathcal{N}$ such that the set $F(x)$ is closed for all $x \in \mathcal{C}$ and the set of points of continuity of F is analytic and not Borel. Moreover the set F is a Borel subset of $\mathcal{C} \times \mathcal{N}$.*
- (b) *There is a multi-valued function $F : [0, 1] \Rightarrow [0, 1]$ for which the set of the points of continuity of F is analytic and not Borel. Moreover the set F is a Borel subset of $[0, 1] \times [0, 1]$.*

It is perhaps useful to make the following remarks. If we replace in (a) of Theorem 2.1 the condition about $F(x)$ being compact for all x with “ $F(x)$ is closed for all x ”, then from (a) of Theorem 2.8 we can see that the result fails in the worst possible way. Also -in connection with (b) of Theorem 2.1- we can see that if we drop the hypothesis about Y being exhaustible by compact sets but keep the second condition “ $F(x)$ is closed for all x ”, then again the result fails in the worst possible way.

If we replace in (a) of Theorem 2.1 the hypothesis “ $F(x)$ is compact for all x ”, with “ Y is compact” then still the result fails in the worst possible way.

In conclusion if we want to obtain that the set of points of continuity of a multi-valued function F is Borel, then we cannot drop the condition “ $F(x)$ is closed for all x ”. But yet this condition alone is not sufficient in order to derive this result as long as Y is neither compact nor exhaustible by compact sets.

Below we give a brief sketch of the proof of the latter theorem.

Sketch of the proof. Let Tr be the set of all (non-empty) trees on ω , (see the Introduction). As we mentioned before the set Tr can be regarded as a compact subspace of the Cantor space \mathcal{C} . From Lemma 2.3 it is enough to construct a multi-valued function $F : Tr \Rightarrow \mathcal{N}$ such that the set of points of continuity of F is not Borel and the set $F(T)$ is closed for all $T \in Tr$.

Denote by IF the set of all ill founded trees i.e., the set of all $T \in Tr$ for which the body $[T]$ is not empty. It is well known (see [4] 27.1) that the set IF is an analytic subset of Tr which is not Borel.² For $T \in Tr$ we define the tree

$$T^{+1} = \{(u(0) + 1, \dots, u(n-1) + 1) \mid u \in T, lh(u) = n\}.$$

²A classical way for proving that a given set $A \subseteq \mathcal{X}$ is not Borel is finding a Borel function $\pi : Tr \rightarrow \mathcal{X}$ such that $IF = \pi^{-1}[A]$. If A was a Borel set then IF would be Borel, a contradiction.

Also we define the set $trm(T)$ as the set of all *terminal* nodes of T i.e., the set of all those u 's in T for which there is no $w \in T$ such that $u \sqsubseteq w$ and $u \neq w$. Define the multi-valued function $F : Tr \Rightarrow \mathcal{N}$ as follows

$$F(T) = [T^{+1}] \cup \{u^\wedge(0, 0, 0, \dots) \mid u \in trm(T^{+1})\}$$

for all $T \in Tr$.

Then we prove that (1) the set F is a Borel subset of $\mathcal{C} \times \mathcal{N}$, (2) for all $T \in Tr$ the set $F(T)$ is a closed subset of the Baire space \mathcal{N} and (3) the multi-valued function F is continuous at T if and only if $T \in IF$. The second assertion of the theorem follows from the first one and Lemmas 2.3 and 2.5.

Question 1. Suppose that we are given a multi-valued function $F : X \Rightarrow Y$ for which we have that the set $F(x)$ is closed for all x and Y is separable. As we have proved before in case where Y is exhaustible by compact sets the set of points of continuity of F is Σ_3^0 and in case where $Y = \mathcal{N}$ it is possible that the latter set is not even Borel. In fact one can see that the latter is true not just for $Y = \mathcal{N}$ but also in case where \mathcal{N} is topologically isomorphic with a closed subset of Y . The question is what happens when Y falls in neither of the previous cases i.e., Y is neither exhaustible by compact sets nor it contains \mathcal{N} as a closed subset. An interesting class of such examples is the class of infinite dimensional separable *Banach spaces* i.e., (infinite dimensional) linear normed spaces which are complete and separable under that norm. Any such space is not exhaustible by compact sets and it does not contain \mathcal{N} as a closed subset. Therefore the theorems of this article provide no information in this case. It would be interesting to find the best upper bound for the complexity of the set of points of continuity of F when Y is an infinite dimensional separable Banach space and the set $F(x)$ is closed for all x .

3 Strong Continuity.

We continue with some results regarding the set of points of *strong* continuity of a multi-valued function F . In particular we will prove the corresponding of Theorem 2.1 and Proposition 2.7. The existence of examples which show that these results are optimum is still a subject under investigation.

Let us begin with some remarks. As we mentioned in the beginning, Theorem 2.1 does not require any additional hypothesis about F as a subset of $X \times Y$. However the following remark suggests that this is not the case for strong continuity.

Remark 3.1. Let A be a dense subset of $[0, 1]$; define the multi-valued function $F : [0, 1] \Rightarrow \{0, 1\}$ as follows

$$F(x) = \{0\}, \text{ if } x \in A \text{ and } F(x) = \{0, 1\} \text{ if } x \notin A,$$

for all $x \in [0, 1]$. We claim that the set of points of strong continuity of F is exactly the set A . Let $x \in A$, $y \in F(x)$ and $\varepsilon > 0$. Take $\delta = 1 > 0$ and let $x' \in (x - \delta, x + \delta)$. We have that $y = 0$ and also since $0 \in F(x')$ we can take $y' = 0$; so $|y - y'| = 0 < \varepsilon$. Now let $x \notin A$. We take $y = 1 \in F(x)$ and $\varepsilon = \frac{1}{2}$. Let any $\delta > 0$. Since A is a dense subset of $[0, 1]$ there is some $x' \in A$ such that $x' \in (x - \delta, x + \delta)$. Clearly for all $y' \in F(x')$ we have that $y' = 0$ and so $|y - y'| = 1 > \varepsilon$.

Since there are dense subsets of $[0, 1]$ which are way above the level of analytic sets from Remark 3.1 we can see that there is no hope to obtain the corresponding of Theorem 2.1 without any additional assumptions about the complexity of the set F . Notice also that those assumptions about the set F have to be at least as strong as the result that we want to derive. For example it is well known that there is a dense Π_3^0 set $A \subseteq [0, 1]$ which is not Σ_3^0 ; hence by taking the multi-valued function F of Remark 3.1 with respect to that set A we can see that F is Δ_4^0 as a subset of $[0, 1] \times [0, 1]$ and that the set of points of strong continuity of F (i.e., the set A) is not Σ_3^0 . In other words if we want to result to a Σ_3^0 set we need to

assume that F does not go above the third level of the Borel hierarchy. The following may be regarded as the corresponding strong-continuity analogue of Theorem 2.1.

Theorem 3.2. *Let (X, p) and (Y, d) be metric spaces with (Y, d) being separable and let $F : X \Rightarrow Y$ be a multi-valued function such that F is a Σ_2^0 subset of $X \times Y$.*

- (a) *If Y is compact and the set $F(x)$ is closed for all $x \in X$ then the set of points of strong continuity of F is Π_2^0 .*
- (b) *If Y is exhaustible by compact sets and the set $F(x)$ is closed for all $x \in X$, then the set of points of strong continuity of F is Σ_3^0 .*

We continue with the corresponding of Proposition 2.7.

Proposition 3.3. *Let (X, p) and (Y, d) be complete and separable metric spaces and let $F : X \Rightarrow Y$ be a multi-valued function such that the set $F \subseteq X \times Y$ is analytic. Then the set of points of strong continuity of F is co-analytic.*

We conclude this article with some remarks which concern all previous results. The author would like to thank the anonymous referee for raising the questions stated below.

Remark 3.4. All results above are in the context of classical descriptive set theory. One could ask whether the corresponding results are also true in the context of *effective* descriptive set theory. In the latter context one deals with the notion of a recursive function $f : \omega^k \rightarrow \omega$ and of a recursive subset of ω^k . We assume that our given metric space (X, d) is complete, separable and that there is a countable dense sequence $\{r_i \mid i \in \omega\}$ such that the relations $d(r_i, r_j) < q$, $d(r_i, r_j) \leq q$ for $i, j \in \omega$ and $q \in \mathbb{Q}^+$, are recursive. (An example of such space is \mathbb{R} with $\{r_i \mid i \in \omega\} = \mathbb{Q}$.) One takes then the family $\{N(X, s) \mid s \in \omega\}$ of all open balls with centers from the set $\{r_i \mid i \in \omega\}$ and rational radii and defines the class of *semirecursive* sets or “effectively open” sets as the sets which are recursive unions of sets of the form $N(X, s)$. The analogous notions go through the whole hierarchy of Borel and analytical sets i.e., one constructs the family of effectively closed, effectively G_δ , effectively analytic sets and so on. The latter classes of sets are also called *lightface* classes. The usual inclusion properties hold also for the lightface classes. For example every effectively closed set is effectively G_δ . We should point out that there are only countably many subsets of a fixed space X which belong to a specific lightface class. Also all singletons $\{q\}$ with $q \in \mathbb{Q}$ belong to every one of the lightface classes mentioned above except from the one of semi-recursive sets. The reader can refer to [5] for a detailed exposition of this theory. One natural question which arises is if the results which are presented in this article hold in the context of effective descriptive set theory. For example: if $F : \mathbb{R} \Rightarrow \mathbb{R}$ is a bounded multi-valued function such that the set $F(x)$ is effectively closed, is it true that the set of points of continuity of F is effectively G_δ ? As the next proposition shows the answer to this question is negative even if F is a single-valued function.

Let us say that a family of sets Γ is *closed under negation* if whenever $A \subseteq X$ is in Γ then $X \setminus A$ is in Γ as well.

Proposition 3.5. *Suppose that Γ is a class of sets which is closed under negation and the family $\{A \subseteq \mathbb{R} \mid A \in \Gamma\}$ is countable. Then there is a function $f : \mathbb{R} \rightarrow \{\frac{1}{n+1} \mid n \in \omega\} \cup \{0\}$ such that the set of points of continuity of f is not a member of Γ . In particular (by choosing Γ as the lightface Δ_2^1 class) there is a function $f : \mathbb{R} \rightarrow [0, 1]$ such that the singleton $\{f(x)\}$ is effectively closed for all $x \in \mathbb{R}$ but the set of points of continuity of f is neither effectively analytic nor effectively co-analytic.*

Question 2. In case we take Γ to be the lightface Δ_n^0 class for some small $n \in \omega$, it would be interesting to see whether one can construct a function f which satisfies the first conclusion of the previous proposition and has the additional property that the graph of f belongs to Γ .

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